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FACTORING NONNIL IDEALS INTO PRIME AND INVERTIBLE IDEALS

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Abstract

For a commutative ring R, let $\mathrm{Nil}(R)$ be the set of all nilpotent elements of R, $Z(R)$ the set of all zero divisors of R, and $T(R)$ the total quotient ring of R. Set $\mathcal{H} = \{R \mid R$ is a commutative ring and $\mathrm{Nil}(R)$ is a divided prime ideal of $R\}$. For a ring $R \in \mathcal{H}$, let $\phi : T(R) \longrightarrow R_{\mathrm{Nil}(R)}$ be such that $\phi(a/b) = a/b$ for every $a \in R$ and $b \in R \backslash Z(R)$. A ring R is called a *ZPUI ring* if every proper ideal of R can be written as a finite product of invertible and prime ideals of R. This paper gives a generalization of the concept of ZPUI domains (which was extensively studied by Olberding) to the context of rings that are in the class $\mathcal{H}$. Let $R \in \mathcal{H}$. If every nonnil ideal of R can be written as a finite product of invertible and prime ideals of R; then R is called a *nonnil ZPUI ring*; also, if every nonnil ideal of $\phi(R)$ can be written as a finite product of invertible and prime ideals of $\phi(R)$, then R is called a *nonnil ϕ-ZPUI ring*. The theory of ϕ-ZPUI rings is shown here to resemble that of ZPUI domains.

1. *Introduction*

We assume throughout that all rings are commutative with $1 \neq 0$. Let R be a ring. Then $T(R)$ denotes the total quotient ring of R, $Z(R)$ denotes the set of zero divisors of R, and $\mathrm{Nil}(R)$ denotes the set of nilpotent elements of R. The elements in $R \setminus Z(R)$ are referred to as *regular elements*, and an ideal I is said to be regular if it contains at least one regular element. For a nonzero ideal I, regular or not, we let $I^{-1} = \{x \in T(R) \mid xI \subset R\}$. An ideal I of a ring R is called *invertible* if $II^{-1} = R$.

We start by recalling some background material. Recall from [**12**] and [**5**] that a prime ideal P of R is called a *divided prime* if $P \subset (x)$ for every $x \in R \setminus P$; thus a divided prime ideal is comparable to every ideal of R. In [**4**, **6**–**9**], the author investigated the class of rings $\mathcal{H} = \{R \mid R$ is a commutative ring and $\mathrm{Nil}(R)$ is a divided prime ideal of $R\}$. Also, Anderson, Lucas and the author have undertaken further investigations into the class $\mathcal{H}$ in [**2**, **3**] and, most recently, [**10**].

In this paper, we take the concept of the factorization of ideals of an integral domain into a finite product of invertible and prime ideals, which was previously extensively studied by Olberding [**21**], and we generalise it to the context of rings that are in the class $\mathcal{H}$. Observe that if R is an integral domain, then $R \in \mathcal{H}$. An ideal I of a ring R is said to be a *nonnil ideal* if $I \not\subseteq \mathrm{Nil}(R)$. For each $R \in \mathcal{H}$, the map $\phi : T(R) \longrightarrow R_{\mathrm{Nil}(R)}$, defined by $\phi(a/b) = a/b$ for each $a \in R$ and $b \in R \backslash Z(R)$, was introduced by the author in [**4**]. The map ϕ is a ring homomorphism from $T(R)$ into $R_{\mathrm{Nil}(R)}$, and ϕ restricted to R is a ring homomorphism from R into $R_{\mathrm{Nil}(R)}$ given by $\phi(x) = x/1$ for each $x \in R$. Note that $T(\phi(R)) = R_{\mathrm{Nil}(R)}$. Let $R \in \mathcal{H}$. Then R is said to be a *ϕ-ZPUI ring* if each nonnil ideal I of $\phi(R)$ can be written as $I = JP_1P_2 \ldots P_n$, where J is an invertible ideal of $\phi(R)$ and $P_1, P_2, \ldots, P_n$ are

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prime ideals of $\phi(R)$. If every nonnil ideal I of R can be written as $I = JP_1P_2 \dots P_n$, where J is an invertible ideal of R and $P_1, P_2, \dots, P_n$ are prime ideals of R, then R is said to be a *nonnil ZPUI ring*. Commutative ϕ-ZPUI rings that are in $\mathcal{H}$ are characterized in Theorem 2.9. Examples of ϕ-ZPUI rings that are not ZPUI rings are constructed in Theorem 2.13. It is shown in Theorem 2.14 that a ϕ-ZPUI ring is the pullback of a ZPUI domain. It is shown in Theorem 3.1 that a nonnil ZPUI ring is a ϕ-ZPUI ring. Examples of ϕ-ZPUI rings that are not nonnil ZPUI rings are constructed in Theorem 3.2.

If $\mathrm{Nil}(R)$ is divided, then it is also the nilradical of $T(R)$, and the kernel of the map ϕ is also a common ideal of R and $T(R)$. Other useful features of each ring $R \in \mathcal{H}$ (see [**4**]) include the following:

(i) $\phi(R) \in \mathcal{H}$;
(ii) $T(\phi(R)) = R_{\mathrm{Nil}(R)}$ is quasilocal with maximal ideal $\mathrm{Nil}(\phi(R))$;
(iii) $\phi(R)$ is naturally isomorphic to $R/\mathrm{Ker}(\phi)$;
(iv) $\mathrm{Nil}(R_{\mathrm{Nil}(R)}) = \phi(\mathrm{Nil}(R)) = \mathrm{Nil}(\phi(R)) = Z(\phi(R))$; and
(v) $R_{\mathrm{Nil}(R)}/\mathrm{Nil}(\phi(R)) = T(\phi(R))/\mathrm{Nil}(\phi(R))$ is the quotient field of $\phi(R)/\mathrm{Nil}(\phi(R))$.

If I is a nonnil ideal of a ring $R \in \mathcal{H}$, then observe that $\mathrm{Nil}(R) \subset I$.

Throughout the paper we will use the technique of the idealization of a module to construct examples. Recall that for an R-module B, the idealization of B over R is the ring formed from $R \times B$ by defining addition and multiplication as $(r, a)+(s, b) = (r + s, a + b)$ and $(r, a)(s, b) = (rs, rb + sa)$, respectively. A standard notation for the 'idealized ring' is $R(+)B$. See [**17**–**19**] for the basic properties of these rings. For further background material, we recommend the papers [**1**, **11**, **14**, **15**].

2. ϕ-ZPUI RINGS

We recall the following two lemmas from [**2**], which will allow us to prove the theorem that follows them.

LEMMA 2.1 [**2**, Lemma 2.3]. *Let $R \in \mathcal{H}$ with $\mathrm{Nil}(R) = Z(R)$, and let I be an ideal of R. Then I is an invertible ideal of R if and only if $I/\mathrm{Nil}(R)$ is an invertible ideal of $R/\mathrm{Nil}(R)$.*

LEMMA 2.2 [**2**, Lemma 2.5]. *Let $R \in \mathcal{H}$, and let P be a prime ideal of R. Then R/P is ring-isomorphic to $\phi(R)/\phi(P)$. In particular, $R/\mathrm{Nil}(R)$ is ring-isomorphic to $\phi(R)/\mathrm{Nil}(\phi(R))$.*

THEOREM 2.3. *Let $R \in \mathcal{H}$. Then R is a ϕ-ZPUI ring if and only if $R/\mathrm{Nil}(R)$ is a ZPUI domain.*

Proof. Suppose that R is a ϕ-ZPUI ring. Set $D = \phi(R)/\mathrm{Nil}(\phi(R))$, and let L be a nonzero ideal of D. Then $L = I/\mathrm{Nil}(\phi(R))$ for some nonnil ideal I of $\phi(R)$. Thus $I = JP_1P_2 \dots P_n$, where J is an invertible ideal of $\phi(R)$ and $P_1, P_2, \dots, P_n$ are prime ideals of $\phi(R)$. Since $\mathrm{Nil}(\phi(R)) = Z(\phi(R))$, we conclude that $J/\mathrm{Nil}(\phi(R))$ is an invertible ideal of D, by Lemma 2.1. Thus

$$L = I/\mathrm{Nil}(\phi(R)) = (J/\mathrm{Nil}(\phi(R)))(P_1/\mathrm{Nil}(\phi(R))) \dots (P_n/\mathrm{Nil}(\phi(R))),$$

and hence D is a ZPUI domain. Since D is ring-isomorphic to $R/\operatorname{Nil}(R)$ by Lemma 2.2, we conclude that $R/\operatorname{Nil}(R)$ is a ZPUI domain.

Conversely, suppose that $R/\operatorname{Nil}(R)$ is a ZPUI domain. Then $D = \phi(R)/\operatorname{Nil}(\phi(R))$ is a ZPUI domain, by Lemma 2.2. Let I be a nonnil ideal of $\phi(R)$. Since $\phi(R) \in \mathcal{H}$, $I/\operatorname{Nil}(\phi(R))$ is a nonzero ideal of D. Thus

$$I/\operatorname{Nil}(\phi(R)) = (J/\operatorname{Nil}(\phi(R)))(P_1/\operatorname{Nil}(\phi(R)))\dots(P_n/\operatorname{Nil}(\phi(R))),$$

where J is an invertible ideal of $\phi(R)$ (by Lemma 2.1) and $P_1, P_2, \dots, P_n$ are prime ideals of $\phi(R)$. We show that $I = JP_1P_2\dots P_n$. This follows since $\operatorname{Nil}(\phi(R)) \subset I$ because $\operatorname{Nil}(\phi(R)) \subset P_i$ for each i and $\operatorname{Nil}(\phi(R))$ is a divided prime ideal of $\phi(R)$. Thus R is a ϕ-ZPUI ring. □

LEMMA 2.4. *Let $R \in \mathcal{H}$, and let I be a nonnil ideal of R. Then I is a finitely generated ideal of R if and only if $I/\operatorname{Nil}(R)$ is a finitely generated ideal of $R/\operatorname{Nil}(R)$.*

Proof. The proof is similar to the proof of [**9**, Theorem 2.2]. Suppose that I is a nonnil finitely generated ideal of R. Since $\operatorname{Nil}(R) \subset I$, it is clear that $I/\operatorname{Nil}(R)$ is a finitely generated ideal of $R/\operatorname{Nil}(R)$. Conversely, suppose that $J = I/\operatorname{Nil}(R)$ is a finitely generated ideal of $R/\operatorname{Nil}(R)$. Then $J = (i_1 + \operatorname{Nil}(R), \dots, i_n + \operatorname{Nil}(R))$ for some values of i_m in I. Since $\operatorname{Nil}(R)$ is divided, we may assume that all the i_m are nonnilpotent elements of R, and thus $\operatorname{Nil}(R) \subset (i_1)$. Now let x be a nonnilpotent element of I. Then $x + \operatorname{Nil}(R) = c_1i_1 + \dots + c_ni_n + \operatorname{Nil}(R)$ in $R/\operatorname{Nil}(R)$ for some values of c_m in R. Hence there is a $w \in \operatorname{Nil}(R)$ such that $x + w = c_1i_1 + \dots + c_ni_n$ in R. Since $x \in I \setminus \operatorname{Nil}(R)$, $x \mid w$ in R. Thus $w = xf$ for some $f \in \operatorname{Nil}(R)$. Hence

$$x + w = x + xf = x(1+f) = c_1i_1 + \dots + c_ni_n \qquad \text{in } R.$$

Since $f \in \operatorname{Nil}(R)$, $1+f$ is a unit of R. Thus $x \in (i_1, \dots, i_n)$, and hence I is a finitely generated ideal of R. □

Recall from [**19**] that a ring R is called a *Prüfer ring* if every finitely generated regular ideal of R is invertible. A Prüfer domain R is called a *strongly discrete Prüfer domain*, as in [**20**, **21**], if R has no nonzero prime ideals P such that $P^2 = P$. A ring $R \in \mathcal{H}$ is said to be a *ϕ-Prüfer ring*, as in [**2**], if $\phi(R)$ is a Prüfer ring. We call a ring $R \in \mathcal{H}$ a *nonnil strongly discrete ring* if R has no nonnil prime ideal P such that $P^2 = P$. An integral domain R is called *h-local*, as in [**20**], if each nonzero ideal of R is contained in at most finitely many maximal ideals of R and each nonzero prime ideal P of R is contained in a unique maximal ideal of R. A ring $R \in \mathcal{H}$ is said to be *nonnil h-local* if each nonnil ideal of R is contained in at most finitely many maximal ideals of R and each nonnil prime ideal P of R is contained in a unique maximal ideal of R.

The reader can easily verify the following two lemmas.

LEMMA 2.5. *Let $R \in \mathcal{H}$. Then R is a nonnil h-local ring if and only if $R/\operatorname{Nil}(R)$ is an h-local domain.*

LEMMA 2.6. *Let $R \in \mathcal{H}$. Then R is a nonnil strongly discrete Prüfer ring if and only if $R/\operatorname{Nil}(R)$ is a strongly discrete Prüfer domain.*

We recall the following result from [**2**].

PROPOSITION 2.7 [**2**]. *Let $R \in \mathcal{H}$. Then R is a ϕ-Prüfer ring if and only if $R/\operatorname{Nil}(R)$ is a Prüfer domain.*

Combining Lemmas 2.5 and 2.6 with Proposition 2.7 yields the following result.

PROPOSITION 2.8. *Let $R \in \mathcal{H}$. Then R is a nonnil strongly discrete nonnil h-local ϕ-Prüfer ring if and only if $R/\operatorname{Nil}(R)$ is a strongly discrete h-local Prüfer domain.*

Since the class of integral domains is a subset of $\mathcal{H}$, the following result is a generalization of [**21**, Theorem 2.3].

THEOREM 2.9. *Let $R \in \mathcal{H}$. Then the following statements are equivalent.*

(1) R is a ϕ-ZPUI ring.

(2) Every nonnil proper ideal of R can be written as a product of prime ideals of R and a finitely generated ideal of R.

(3) Every nonnil proper ideal of $\phi(R)$ can be written as a product of prime ideals of $\phi(R)$ and a finitely generated ideal of $\phi(R)$.

(4) R is a nonnil strongly discrete nonnil h-local ϕ-Prüfer ring.

Proof. Set $D = R/\operatorname{Nil}(R)$.

(1) $\Longrightarrow$ (2). Since R is a ϕ-ZPUI ring, D is a ZPUI domain, by Theorem 2.3. Let I be a nonnil proper ideal of R. Then, by [**21**, Theorem2.3], we have $I/\operatorname{Nil}(R) = (J/\operatorname{Nil}(R))(P_1/\operatorname{Nil}(R))\ldots(P_n/\operatorname{Nil}(R))$, where J is a (nonnil) finitely generated ideal of R (by Lemma 2.4) and $P_1, P_2, \ldots, P_n$ are prime ideals of R. Since $\operatorname{Nil}(R)$ is divided, it is easily verified that $I = JP_1 \ldots P_n$.

(2) $\Longrightarrow$ (3). Let L be a nonnil proper ideal of $\phi(R)$. Then $L = \phi(I)$ for some nonnil proper ideal I of R. Since $I = JP_1 \ldots P_n$, where J is a (nonnil) finitely generated ideal of R and $P_1, P_2, \ldots, P_n$ are prime ideals of R, it is easily verified that $L = \phi(I) = \phi(J)\phi(P_1)\ldots\phi(P_n)$, where $\phi(J)$ is a finitely generated ideal of $\phi(R)$ and $\phi(P_1), \ldots, \phi(P_n)$ are (nonnil) prime ideals of $\phi(R)$.

(3) $\Longrightarrow$ (4). Let $F = \phi(R)/\operatorname{Nil}(\phi(R))$. Then every nonzero ideal of F can be written as a product of prime ideals of F and a finitely generated ideal of F, and thus F is a strongly discrete h-local Prüfer domain, by [**21**, Theorem 2.3]. Since F is ring-isomorphic to D, we conclude that D is a strongly discrete h-local Prüfer domain, and hence R is a nonnil strongly discrete nonnil h-local ϕ-Prüfer ring, by Proposition 2.8.

(4) $\Longrightarrow$ (1). Since R is a nonnil strongly discrete nonnil h-local ϕ-Prüfer ring, we conclude that $D = R/\operatorname{Nil}(R)$ is a strongly discrete h-local Prüfer domain, by Proposition 2.8. Thus D is a ZPUI domain, by [**21**, Theorem 2.3], and hence R is a ϕ-ZPUI ring, by Theorem 2.3. □

Let $R \in \mathcal{H}$ such that $Z(R) = \operatorname{Nil}(R)$. Then $\phi(R) = R$, and hence R is a ϕ-ZPUI ring if and only if R is a nonnil ZPUI ring. We state this connection in the following corollary.

COROLLARY 2.10. *Let $R \in \mathcal{H}$ such that $\operatorname{Nil}(R) = Z(R)$. The following statements are equivalent.*

(1) *R is a nonnil ZPUI ring.*
(2) *R is a ϕ-ZPUI ring.*
(3) *Every nonnil proper ideal of R can be written as a product of prime ideals of R and a finitely generated ideal of R.*
(4) *R is a nonnil strongly discrete nonnil h-local Prüfer ring.*

Recall that a *special primary ring* is a quasilocal commutative ring R with maximal ideal M such that every proper ideal of R is a power of M. We state the following useful lemma.

LEMMA 2.11 (see [**21**, Lemma 3.2 and Theorem 3.3]). *Let $R \in \mathcal{H}$. Then R is a ZPUI ring if and only if R is either a strongly discrete h-local Prüfer domain, or a special primary ring.*

Proof. Suppose that R is a ZPUI ring. First observe that if a ring $A \cong A_1 \oplus \ldots \oplus A_n$ (where each A_i is a ring with $1 \neq 0$) and $n \geqslant 2$, then Nil(A) is never divided, and hence $A \notin \mathcal{H}$. Now, since R is a ZPUI ring, by [**21**, Theorem 3.3] we have $R \cong D_1 \oplus \ldots \oplus D_n$, where each D_i is either a strongly discrete h-local Prüfer domain or a special primary ring. Since Nil(R) is divided, by the observation that we have just stated we conclude that $n = 1$, and thus R is either a strongly discrete h-local Prüfer domain, or a special primary ring. The converse is clear, by [**21**, Theorem 3.3]. □

Our non-domain examples of ϕ-ZPUI rings that are not ZPUI rings are provided by the idealization construction $R(+)B$ arising from a ring R and an R-module B as in [**19**, Chapter VI]. We recall this construction. Let $R(+)B = R \times B$, and define:
(i) $(r, b) + (s, c) = (r + s, b + c)$;
(ii) $(r, b)(s, c) = (rs, sb + rc)$.
Under these definitions, $R(+)B$ becomes a commutative ring with identity. We recall the following two facts.

PROPOSITION 2.12. *Let R be a ring, B an R-module, and $Z(B)$ the set of zero divisors on B. Then the following statements hold.*
(1) [**19**, Theorem 25.1]: *The ideal J of $R(+)B$ is prime if and only if $J = P(+)B$, where P is a prime ideal of R. Likewise, the ideal J of $R(+)B$ is maximal if and only if $J = P(+)B$, where P is a maximal ideal of R. Hence the Krull dimension of R is equal to the Krull dimension of $R(+)B$.*
(2) [**19**, Theorem 25.3]: *$(r, b) \in Z(R(+)B)$ if and only if $r \in Z(R) \cup Z(B)$.*

Olberding showed in [**21**, Corollary 2.4] that for each $n \geqslant 1$, there exists a ZPUI domain with Krull dimension n. A Dedekind domain is a trivial example of a ZPUI domain.

THEOREM 2.13. *Let A be a ZPUI domain (that is, A is a strongly discrete h-local Prüfer domain, by* [**21**, Theorem 2.3]*) with Krull dimension $n \geqslant 1$ and quotient field F, and let K be an extension ring of F (that is, K is a ring and $F \subseteq K$). Then $R = A(+)K \in \mathcal{H}$ is a ϕ-ZPUI ring with Krull dimension n that is not a ZPUI ring.*

Proof. It is easy to see that $\mathrm{Nil}(R) = \{0\}(+)K$. We show that $\mathrm{Nil}(R)$ is divided. Let $(0,k) \in R$, and let $(a,b) \in R \setminus \mathrm{Nil}(R)$. Then $a \neq 0$, and hence $(0,k) = (a,b)(0,k/a)$. Observe that $k/a \in K$ because $F \subseteq K$. Thus $R \in \mathcal{H}$. R is not a ZPUI ring, by Lemma 2.11. Since $R/\mathrm{Nil}(R) \cong A$ is a ZPUI domain, we see that R is a ϕ-ZPUI ring, by Theorem 2.3. The Krull dimension of R is n, by Proposition 2.12(1). □

In the following theorem, we show that a ϕ-ZPUI ring is a pullback of a ZPUI domain. A good reference for pullback is the article of Fontana [**13**].

THEOREM 2.14. *Let $R \in \mathcal{H}$. Then R is a ϕ-ZPUI ring if and only if $\phi(R)$ is ring-isomorphic to a ring A obtained from the following pullback diagram, where T is a zero-dimensional quasilocal ring with maximal ideal M, A/M is a ZPUI ring that is a subring of T/M, the vertical arrows are the usual inclusion maps, and the horizontal arrows are the usual surjective maps.*

$$\begin{array}{ccc} S & \longrightarrow & A/M \\ \downarrow & & \downarrow \\ T & \longrightarrow & T/M \end{array}$$

Proof. Suppose that $\phi(R)$ is ring-isomorphic to a ring A obtained from the diagram given in the theorem. Then $A \in \mathcal{H}$ and $\mathrm{Nil}(A) = Z(A) = M$. Since A/M is a ZPUI domain, A is a ϕ-ZPUI ring, by Theorem 2.3, and thus R is a ϕ-ZPUI ring.

Conversely, suppose that R is a ϕ-ZPUI ring. Then, letting $T = R_{\mathrm{Nil}(R)}$, $M = \mathrm{Nil}(R_{\mathrm{Nil}(R)})$, and $A = \phi(R)$ yields the desired pullback diagram. □

3. *Nonnil ZPUI rings and nonnil ZPI rings*

We start with the following result.

THEOREM 3.1. *Let $R \in \mathcal{H}$ be a nonnil ZPUI ring. Then R is a ϕ-ZPUI ring, and hence all the following statements hold.*

(1) *$R/\mathrm{Nil}(R)$ is a ZPUI domain.*

(2) *Every nonnil proper ideal of R can be written as a product of prime ideals of R and a finitely generated ideal of R.*

(3) *Every nonnil proper ideal of $\phi(R)$ can be written as a product of prime ideals of $\phi(R)$ and a finitely generated ideal of $\phi(R)$.*

(4) *R is a nonnil strongly discrete nonnil h-local ϕ-Prüfer ring.*

(5) *R is a nonnil strongly discrete nonnil h-local Prüfer ring.*

Proof. Let L be a nonnil proper ideal ideal of $\phi(R)$. Then $L = \phi(I)$ for some nonnil proper ideal I of R. Since $I = JP_1P_2 \ldots P_n$, where J is an invertible ideal of R and $P_1, P_2, \ldots, P_n$ are prime ideals of R, it follows that

$$L = \phi(I) = \phi(J)\phi(P_1)\ldots\phi(P_n),$$

where $\phi(J)$ is an invertible ideal of $\phi(R)$ and $\phi(P_1), \phi(P_2), \ldots, \phi(P_n)$ are prime ideals of $\phi(R)$. Thus R is a ϕ-ZPUI ring.

Now statement (1) is clear by Theorem 2.3, and statements (2), (3) and (4) are clear from Theorem 2.9. For statement (5), by [**2**] just observe that R is a Prüfer ring because R is a ϕ-Prüfer ring. □

In the following result, we show that if $R \in \mathcal{H}$ is a ϕ-ZPUI ring, then R does not need to be a nonnil ZPUI ring. In particular, we show that if $R \in \mathcal{H}$ satisfies any of the five statements in Theorem 3.1, then R does not need to be a nonnil ZPUI ring.

THEOREM 3.2. *Let A be a ZPUI domain that is not a Dedekind domain, with Krull dimension $n \geqslant 1$ and quotient field K. Then*

$$R = A(+)K/A \in \mathcal{H}$$

is a ϕ-ZPUI ring, with Krull dimension n, that is not a nonnil ZPUI ring.

Proof. Since $\mathrm{Nil}(R) = \{0\}(+)K/A$, by a similar calculation to that given in the proof of Theorem 2.13, we conclude that $\mathrm{Nil}(R)$ is divided, and thus $R \in \mathcal{H}$. Since $R/\mathrm{Nil}(R) \cong A$ is a ZPUI domain, R is a ϕ-ZPUI ring by Theorem 2.3, and the Krull dimension of R is n by Proposition 2.12(1). Since every non-unit element of R is a zero divisor of R by Proposition 2.12(2), we conclude that $T(R) = R$, and thus R is the only invertible ideal of R. Suppose that R is a nonnil ZPUI ring. Then every nonnil proper ideal of R is a finite product of prime ideals of R, and hence every proper ideal of the integral domain $R/\mathrm{Nil}(R) \cong A$ is a finite product of prime ideals of $R/\mathrm{Nil}(R)$. Thus $R/\mathrm{Nil}(R) \cong A$ is a Dedekind domain, a contradiction. Hence R is not a nonnil ZPUI ring. □

Recall from [**16**] that a ring R is called a *ZPI ring* if every nonzero proper ideal of R is uniquely a product of prime ideals of R, and R is called a *general ZPI ring* if every nonzero proper ideal of R is a product of prime ideals of R. In [**4**], it is said that a ring $R \in \mathcal{H}$ is a *nonnil ZPI ring* if every nonnil proper ideal of R is uniquely a product of (nonnil) prime ideals of R, and it is said that R is a *general nonnil ZPI ring* if every nonnil proper ideal of R is a product of (nonnil) prime ideals of R. A ring $R \in \mathcal{H}$ is called a *ϕ-Dedekind ring* as in [**4**], if every nonnil ideal of R is invertible. A ring $R \in \mathcal{H}$ is called a *nonnil Noetherian ring* as in [**9**] if every nonnil ideal of R is finitely generated.

We recall the following two results from [**4**].

PROPOSITION 3.3 [**4**, Corollary 2.17]. *Let $R \in \mathcal{H}$. Then the following statements are equivalent.*

(1) *R is a ϕ-Dedekind ring.*
(2) *R is a nonnil ZPI ring.*
(3) *R is a general nonnil ZPI ring.*

PROPOSITION 3.4 [**4**, Proposition 2.11]. *Let $R \in \mathcal{H}$ be a nonnil Noetherian ring. Then R is a ϕ-Dedekind ring if and only if R is a ϕ-Prüfer ring.*

Combining Propositions 3.3 and 3.4 with Theorem 2.9, we arrive at the following result.

Corollary 3.5. *Let $R \in \mathcal{H}$ be a nonnil Noetherian ring. Then the following statements are equivalent.*

(1) *R is a ϕ-ZPUI ring.*
(2) *R is a nonnil ZPUI ring.*
(3) *R is a nonnil ZPI ring.*
(4) *R is a general nonnil ZPI ring.*
(5) *R is a nonnil strongly discrete nonnil h-local ϕ-Prüfer ring.*
(6) *R is a nonnil strongly discretere nonnil h-local ϕ-Dedekind ring.*

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